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Weakly Gibbsian Measures for Lattice Spin Systems

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We present a weaker notion of Gibbs measures by requiring that only an almost everywhere absolutely summable potential is given. This has recently appeared in the context "Gibbsian versus non-Gibbsian measures". We give a first exploration of the main features of this weaker notion. We concentrate on the questions of where do weakly Gibbsian measures appear and what remains of their thermodynamic description.

KEY WORDS: Renormalization group pathologies; Gibbs measures; transformations.

1. INTRODUCTION

Recent years have produced a remarkable effort for understanding the occurrence and properties of non-Gibbsian states, contrasting the earlier held belief that in most cases the relevant states can be described in terms of Gibbs measures. The main sources of non-Gibbsian states are real-space renormalization group theory,[15, 16, 20, 10, 9, 8] interacting particle systems,[22, 28, 37] and other fields of applied probability theory.[2, 12, 27, 28, 31, 36, 19, 32, 17, 11] Recently it was realized that at least for some non-Gibbsian states a Gibbsian description in terms of an interaction potential can be restored. More precisely, in refs. [5, 1, 30] a full measure subset of configurations is constructed on which a potential can be defined.

In this paper we are taking up this approach and we are asking in what generality we can give a working definition of weakly Gibbsian states that are bearing the main features of usual Gibbs measures. First we introduce the notion of weakly Gibbsian measures in a general context. Then we

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will concentrate on non-Gibbsian measures obtained by transforming Gibbs probability measures, and give a criterion for their weak Gibbsianness. We argue, by using stochastic-geometric methods, that the occurrence of weakly Gibbsian states is ubiquitous in typical Pirogov-Sinai situations. We end by discussing the thermodynamic functions and their relation in a variational principle. Also, many open questions are discussed.

2. DEFINITION OF WEAKLY GIBBSIAN MEASURES

On the d-dimensional lattice $\mathbb{Z}^d$ a spin variable $\sigma(x) \in S$ belonging to a fixed finite set S is assigned to each site $x \in \mathbb{Z}^d$. The full configuration space is $\Omega = S^{\mathbb{Z}^d}$ and it consists of all configurations $\sigma = (\sigma(x), x \in \mathbb{Z}^d)$. For an $A \subset \mathbb{Z}^d$ we write $\Omega_A = S^A$, and the restriction of a $\sigma \in \Omega$ to Ω_A is denoted by σ_A. The symbol $\mathscr{E}$ is used for the collection of all finite subsets $B \subset \mathbb{Z}^d$, and we put $|B|$ for the cardinality of B. For $A \subset \mathbb{Z}^d$, $A^c = \mathbb{Z}^d \backslash A$ stands for its complement. Ω is equipped with the product topology and with its associated Borel σ-field $\mathscr{F}$. For $A \subset \mathbb{Z}^d$, $\mathscr{F}_A \subset \mathscr{F}$ is the sub-σ-field of all events occurring in $A \subset \mathbb{Z}^d$ (generated by the functions $\sigma \mapsto \sigma(x)$, $x \in A$). Lattice translations θ_x, $x \in \mathbb{Z}^d$, are first defined on configurations by $(\theta_x \sigma)(y) = \sigma(x+y)$, and their obvious extension to functions on Ω and to probability measures on $(\Omega, \mathscr{F})$ are further also considered.

We consider a non-empty translation invariant subset of configurations $K \subset \Omega$, which we assume to be a tail event (i.e., to be belonging to the tail field $\mathscr{T} = \bigcap_{B \in \mathscr{E}} \mathscr{F}_{B^c}$). We require that the configuration $\eta_A \sigma_{A^c} = (\eta(x), x \in A; \sigma(x), x \in A^c) \in K$, whenever $\sigma \in K$, for all $A \in \mathscr{E}$ and $\eta \in \Omega$. Hence every $\eta \in \Omega$ can be approximated by elements of K (K is not closed if $K \neq \Omega$).

The elements of K will be viewed as typical configurations related to an interaction U given between the spins. This interaction is defined by a family of bounded functions $U_B \colon \Omega_B \to \mathbb{R}$, for $B \in \mathscr{E}$. Each U_B depends only on spins inside B, and we put $U_\varnothing = 0$. We assume that this interaction is translation invariant. More importantly, we assume that the energy of a configuration $\sigma \in K$ due to the interaction of the spin at $x \in \mathbb{Z}^d$ with the rest of the system is bounded: For each $x \in \mathbb{Z}^d$ and $\sigma \in K$, there exist bounds $b_x(\sigma) < \infty$ for which

$$\sum_{B \ni x} |U_B(\sigma)| \leqslant b_x(\sigma) = b_0(\theta_x \sigma) \tag{2.1}$$

These bounds should be thought of as functions growing with the size of the "bad" cluster around $x \in \mathbb{Z}^d$ in which the configuration is "untypical". This will become clearer later on. For the moment it is useful to think of the set K as consisting of configurations which in some sense resemble a reference configuration $\tau \in \Omega$ for which $\sum_{B \ni x} |U_B(\tau)| < \infty$, uniformly in $x \in \mathbb{Z}^d$. We therefore call $\tau \in K$ a "reference configuration" for the interaction $\{U_B\}_{B \in \mathscr{E}}$ if for all $\sigma \in \Omega$ and $\Lambda, B \in \mathscr{E}$ we have

$$|U_B(\sigma_\Lambda \tau_{\Lambda^c})| \leqslant |U_B(\sigma)| \, c(\tau) + c_B(\tau) \tag{2.2}$$

with $c(\tau) < \infty$ and $\sup_{x \in \mathbb{Z}^d} \sum_{B \ni x} c_B(\tau) < \infty$.

The Hamiltonian for a configuration $\sigma \in \Omega$ on $\Lambda \in \mathscr{E}$, and free boundary condition is

$$H_\Lambda(\sigma) = \sum_{B \subset \Lambda} U_B(\sigma) \tag{2.3}$$

It only depends on σ_Λ. Similarly, with boundary condition $\tau \in K$, the Hamiltonian

$$H_\Lambda^\tau(\sigma) = \sum_{B \cap \Lambda \neq \varnothing} U_B(\sigma_\Lambda \tau_{\Lambda^c}) \tag{2.4}$$

stays well defined. In particular, the interaction energy

$$I_{V,W}(\sigma) = H_{V \cup W}(\sigma) - H_V(\sigma) - H_W(\sigma) \tag{2.5}$$

for finite sets $V \cap W = \varnothing$ is bounded by

$$I_{V,V^c}(\sigma_V \tau_{V^c}) \leqslant \sum_{x \in V} b_x^V(\sigma_V \tau_{V^c}) \tag{2.6}$$

with

$$b_x^V(\eta) \geqslant \sum_{\substack{B \ni x \\ B \cap V^c \neq \varnothing}} |U_B(\eta)| \tag{2.7}$$

and $\eta, \tau \in K$.

The finite volume Gibbs measure with respect to the Hamiltonian H_Λ^τ is the probability measure on $(\Omega, \mathscr{F})$ defined by

$$\gamma_\Lambda^\tau(\sigma) = \begin{cases} \dfrac{1}{Z_\Lambda^\tau} \exp[-\beta H_\Lambda^\tau(\sigma)] & \text{if } \sigma_{\Lambda^c} = \tau_{\Lambda^c} \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

Here

$$Z^{\tau}_{\Lambda} = \sum_{\sigma \in \Omega_{\Lambda}} \exp[-\beta H^{\tau}_{\Lambda}(\sigma)] \tag{2.9}$$

is the partition function (normalizing (2.8)), and β is the inverse absolute temperature. Without loss of generality we have chosen the counting measure on S as a priori measure. Note that $\gamma^{\tau}_{\Lambda}(K) = 1$, since $\sigma_{\Lambda} \tau_{\Lambda^c} \in K$ whenever $\tau \in K$.

As usually, the Dobrushin operators γ_{Λ}, $\Lambda \in \mathscr{E}$, can be defined as

$$\begin{aligned} \gamma_{\Lambda}(f)(\eta) &= \gamma^{\eta}_{\Lambda}(f) \\ &= \int \gamma^{\eta}_{\Lambda}(d\sigma)\, f(\sigma) \\ &= \sum_{\sigma \in \Omega_{\Lambda}} f(\sigma_{\Lambda} \eta_{\Lambda^c}) \gamma^{\eta}_{\Lambda}(\sigma) \end{aligned}$$

for all bounded $\mathscr{F}$-measurable functions f on Ω, but only for $\eta \in K$. Clearly, since $\gamma_{\Lambda}(K) = 1$, for every $\Lambda' \subset \Lambda$, $\gamma_{\Lambda} \circ \gamma_{\Lambda'}$ is well defined and the Dobrushin consistency $\gamma_{\Lambda} \circ \gamma_{\Lambda'} = \gamma_{\Lambda}$ holds (i.e., the specification property). Notice that if f is a bounded and local function (that is, it only depends on the spins of a volume Λ), and if $\tau \in K$ is "a reference configuration", then

$$\lim_{n \to \infty} |\gamma_{\Lambda}(f)(\eta^{(n)}) - \gamma_{\Lambda}(f)(\eta)| = 0 \tag{2.10}$$

for all $\eta \in K$, where $\eta^{(n)} = \eta_{\Lambda_n} \tau_{\Lambda^c_n} \in K$ is a sequence of configurations corresponding to an increasing sequence of cubes $\{\Lambda_n\}$ of $\mathbb{Z}^d$. This is what remains of the notion of quasilocality of specifications.[10, 11] Quasilocality is a topological notion and it does not seem compatible with the measure theoretic approach taken here (see also ref. [1]). Nevertheless, one can seek to obtain quasilocality on smaller spaces (in the same way as we have summability on a smaller space here). For details we refer to refs. [25], [24]: pp. 77–80, [11].

Inspired by the above, we introduce the following concept:

Definition 2.1. We say that a probability measure ν on $(\Omega, \mathscr{F})$ is a *weakly Gibbsian measure for the set K and the interaction* $\{U_B\}_{B \in \mathscr{E}}$ if

1. $\nu(K) = 1$
2. $\nu(f) = \int_K \gamma^{\tau}_{\Lambda}(f) \nu(d\tau)$ for all bounded $\mathscr{F}$-measurable functions f and all $\Lambda \in \mathscr{E}$.

Remark 2.1. It does not seem straightforward actually to characterize the $(K, \{U_B\})$ for which an infinite volume weakly Gibbsian state exists. It may happen, for example, that although $\gamma^{\tau}_{\Lambda_n}(K) = 1$ for each $\tau \in K$ and $\Lambda_n \in \mathscr{E}$, we have $(\lim_n \gamma^{\tau}_{\Lambda_n})(K) = 0$. The problem is that an accumulation point of convex combinations of the finite volume Gibbs measures (2.8) might fail to verify condition (1) of Definition 2.1 (condition (2) is obtained by construction).

We have, however, the following result. Denote $\varrho_{\Lambda^c} \circ \gamma_\Lambda = \int_K \varrho_{\Lambda^c}(d\tau)\, \gamma^{\tau}_{\Lambda}$, with some distribution ϱ of boundary conditions.

Proposition 2.1. Suppose $\nu = \lim_n \varrho_{\Lambda_n^c} \circ \gamma^{\tau}_{\Lambda_n}$, $\varrho_{\Lambda_n^c}(K) = 1$, and that there are some $A_{kl}(x)$ in $\mathscr{F}_{\Lambda_k(x)}$ such that

$$K = \bigcup_l \bigcap_{k>l} A_{kl}(x) \tag{2.11}$$

If $\varrho_{\Lambda_n^c} \circ \gamma_{\Lambda_n}(A^c_{kl})$ are summable in k, uniformly in n, then $\nu(K) = 1$ and ν is weakly Gibbsian.

Proof. This follows by

$$\nu(K^c) \leqslant \sum_{k>l} \nu(A^c_{kl}) \to 0 \tag{2.12}$$

as $l \to \infty$. ∎

This is the situation in the examples in refs. [5, 1, 30] (see also (2.16)–(2.18) below, and the set-up of Section 3).

In Preston's book (ref. [33], pp. 33–45) a rather general condition for the existence of an infinite volume Gibbs measure consistent with a given specification was given. Even the structure he used is not general enough for our purposes, and thence his results are not applicable for weakly Gibbsian measures.

Remark 2.2. If in particular $K = \Omega$, i.e. we are dealing with an absolutely summable interaction on the full configuration space, then the weakly Gibbsian measure ν is in fact a Gibbs measure (in the usual sense). Clearly, if $\{U_B\}$ is a uniformly short range interaction, i.e. $U_B = 0$ whenever diam $B > R$ for some $R > 0$, then $K = \Omega$.

Remark 2.3. In terms of conditional distributions, (2) in Definition 2.1 is equivalent to

$$\nu_\Lambda(\cdot \mid \sigma_{\Lambda^c} = \tau_{\Lambda^c}) = \gamma^{\tau}_{\Lambda}(\cdot) \tag{2.13}$$

where the right hand side is defined only for $\tau \in K$. The left hand side is the conditional probability for the weakly Gibbsian measure ν in Λ which is defined almost surely. Therefore the equality holds almost surely.

Remark 2.4. Given a weakly Gibbsian measure ν for $(K, \{U_B\})$ we can ask whether another, "non-equivalent" $(\tilde{K}, \{\tilde{U}_B\})$ exists for which ν is also weakly Gibbsian. It is clear that one can find sometimes a different $\tilde{K} \neq K$ so that ν is weakly Gibbsian both for $(K, \{U_B\})$ and $(\tilde{K}, \{U_B\})$. (For instance, this happens if U_B are invariant under a symmetry operation R, and $\tilde{K} = R(K) \neq K$.) In this case we can take the "largest possible" set K.

Changing the potential is, however, a more subtle problem. It is relevant for the question of multivaluedness of renormalization group transformations (see the first fundamental theorem in ref. [10]). If we know that a transformed measure ν is weakly Gibbsian, the question still remains whether the map $\nu \mapsto \{U_B\}$ is multivalued or not. The broader question is about how to fix the notion of physical equivalence in the weakly Gibbsian context. Our remark here is that it can happen that ν is a weakly Gibbsian measure for some $(K, \{U_B\})$ and at the same time ν is a *bona fide* Gibbs state for an interaction $\{\tilde{U}_B\}$, where $\{U_B\} \in \mathscr{B}_0$, $\{\tilde{U}_B\} \in \mathscr{B}_1$, and $\{U_B\}$ is physically equivalent with $\{\tilde{U}_B\}$ in Ruelle's sense (see p. 929 in [10] for notation and details). As an example, take cubes $\Lambda_n \subset \mathbb{Z}^d$, $d > 2$, of sides $n = 1, 3, 5, \ldots$, centered around the origin, and define for Ising spin variables $\sigma(x) \in \{-1, +1\}$

$$U_B(\sigma) = \begin{cases} \alpha\sigma(x) & \text{if } B = \{x\} \\ \dfrac{1}{n^{2d}} \sum_{x \in B} \sigma(x) & \text{if } B = \Lambda_n + y, \text{ for some } y \in \mathbb{Z}^d \\ & \quad \text{and some } n = 1, 3, 5, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

We can choose α finite such that $\{U_B\}$ is physically equivalent (in Ruelle's sense) to $\{\tilde{U}_B = 0\}$. So the Bernoulli measure ν with $\nu(\sigma(x) = 1) = 1/2$ is an infinite-temperature Gibbs state while

$$\sum_{B \ni 0} |U_B(\sigma)| = \alpha + \sum_{n = 3, 5, \ldots} \frac{1}{n^d} \left| \sum_{x \in \Lambda_n} \sigma(x) \right| \tag{2.15}$$

converges ν-almost surely (for $d > 2$).

Remark 2.5. Remarks 2.1 and 2.4 above are examples of questions asking what remains of the standard theory of Gibbs states in the context of weakly Gibbsian states. Here we give a list of "first", most immediate problems one has to face when taking on the weakly Gibbsian set-up (on the solution of which the proof of the "title statement" of ref. [1] would also depend):

(i) the problem of existence of infinite volume weakly Gibbsian measures (Remark 2.1)

(ii) the problem of physical equivalence of interactions (Remark 2.4)

(iii) equivalence between the Minlos-type definition of an infinite volume weakly Gibbsian state and the DLR-type definition given above (see (2) in Definition 2.1)

(iv) the meaning of a macroscopic state, extremality and phase diagram

(v) when is an interaction weak?; can one make a theory of uniqueness of weakly Gibbsian measures?

(vi) variational principle for weakly Gibbsian measures

(vii) large deviation theory to weakly Gibbsian measures

Remark 2.6. A last remark concerns the relevance of developing a theory of non-Gibbsian states. Should one not be content merely with seeing them appear in physically interesting models? Part of the answer undoubtedly depends on how rich the resulting theory is, that is, to what extent one is able to find a solution to the problems listed in the previous remark. Examples can play an important role here, for instance in suggesting additional natural assumptions on $(K, \{U_B\})$. It may be useful, for example, to assume that

$$K = \bigcup_{i=1}^{n} K_i \tag{2.16}$$

$$K_i = \bigcup_{l} K_i^l(x) \tag{2.17}$$

$$K_i^l(x) = \left\{ \sigma \in \Omega : \forall k \geqslant l \, \frac{1}{|\Lambda_k|} \sum_{y+\Lambda_k \neq x} I(\sigma(y) = \tau^{(i)}(y)) \geqslant 1 - \varepsilon \right\} \tag{2.18}$$

for appropriate $\{\Lambda_n\}_{n=1}^{\infty}, \varepsilon > 0$, and a fixed finite set $\{\tau^{(i)}\}_{i=1}^{n}$ of "reference" configurations ($I(\cdot)$ is the characteristic function). We will see this structure appearing in the following section.

More insight might be obtained from the theory of Gibbs measures for systems of unbounded spins. There too one has to dispose of a set of "bad" configurations (which are growing too fast at infinity). One may ask how to find a correspondence with the notions of superstable and (super)regular interactions in order to find useful and natural conditions on upper bounds of the Hamiltonians and interaction energies (see refs. [34, 35, 21, 23]). In Section 4 below we will investigate some of these points for problem (vii) of Remark 2.5, on what remains of the variational principle.

3. ON A CLASS OF WEAKLY GIBBSIAN STATES

The purpose of this section is to describe a framework in which the notion of weakly Gibbsian states naturally arises, and to give a condition under which weak Gibbsianness holds. In this section we consider a class of weakly Gibbsian measures which appear as transformations of low temperature Gibbs measures.

Consider, for simplicity, a translation invariant nearest neighbour pair interaction U and suppose that $\tau \in \Omega$ is a translation invariant ground state, i.e., for all $\sigma \in \Omega$, $\Lambda \in \mathscr{S}$, $\eta = \sigma_\Lambda \tau_{\Lambda^c}$

$$\mathscr{H}(\sigma \mid \tau) := \sum_{\substack{x, y \in \mathbb{Z}^d \\ |x-y|=1}} [U(\eta(x), \eta(y)) - U(a, a)] \geqslant 0 \tag{3.1}$$

holds, and where we have chosen $\tau(x) = a$. Using this τ as a boundary condition we consider the Gibbs measure $\mu^\tau_{\beta, \Lambda}(\sigma)$ for a finite volume Λ and inverse temperature β. It is known that for a large class of models at low temperatures (within the realm of the Pirogov-Sinai theory[39]), the infinite volume measure μ^τ_β exists independently of the particular subsequence chosen in the thermodynamic limit. We can thus take an increasing sequence of cubes Λ_n centered around the origin and assume that $\mu^\tau_{\beta, \Lambda_n} \to \mu^\tau_\beta$. A stochastic-geometric characterization of the low temperature phase μ^τ_β can be done relatively easily when the low temperature phase is a perturbation of the stable ground state (see e.g. Chapter 18 in ref. [13], and also ref. [14]). When τ is stable, typical configurations of μ^τ_β consist of, at least for sufficiently large β, infinite oceans of τ-valued sites with islands of arbitrary size and shape inside, on which the configuration is different from τ. The μ^τ_β-probability of the origin belonging to such a disagreement cluster of size (say, diameter) k decays exponentially fast as $k \to \infty$, with a rate $m = m(\beta)$ that goes to infinity as the temperature drops to zero. In two dimensions, at least for certain models, this picture remains valid in the whole coexistence region. Again, for details and precise statements we refer to refs. [13], [14], and to [4] in particular for the two dimensional Ising model.

Consider now a sublattice $r\mathbb{Z}^d$ ($r \geqslant 2$) on which we fix an additional internal boundary condition (it can also be seen as a constraint), i.e., we take a $\xi \in \Omega$ and define

$$\mu_{\beta, \Lambda_n}^{\tau, \xi}(\cdot) = \mu_{\beta, \Lambda_n}^{\tau}(\cdot \mid \sigma = \xi \text{ on } \Lambda_n \cap r\mathbb{Z}^d) \tag{3.2}$$

In the same low temperature context as above, it seems natural to expect that the stochastic-geometric characterization of $\mu_{\beta, \Lambda_n}^{\tau, \xi}$ is very much similar to that given above for $\mu_{\beta, \Lambda_n}^{\tau}$, at least when ξ in some sense resembles τ. This last proviso that "ξ resembles τ" is certainly necessary for the statement that "$\mu_{\beta}^{\tau, \xi}$ resembles μ_{β}^{τ}", and we now make this more precise.

Write $\Omega_r := S^{r\mathbb{Z}^d}$ and consider the growing sequence of volumes $V_k = \Lambda_k \cap r\mathbb{Z}^d$, $k = 1, 2, \ldots$, together with their shifts $V_k(x) = (\Lambda_k + x) \cap r\mathbb{Z}^d$, for $x \in r\mathbb{Z}^d$. Remember that $\tau(x) = a$, for all $x \in \mathbb{Z}^d$, with some $a \in S$. The frequency of agreement between $\xi \in \Omega_r$ and τ in V_k is measured by

$$\text{agr}_{k, x}(\xi, \tau) = \frac{1}{|V_k|} \sum_{x \in V_k} I(\xi(x) = a) \tag{3.3}$$

Define now

$$K_l^{\tau}(x) = \{\xi \in \Omega_r : \forall k > l \ \text{agr}_{k, x}(\xi, \tau) > 1 - \varepsilon\} \tag{3.4}$$

$$K^{\tau} = \bigcup_l K_l^{\tau}(x) \tag{3.5}$$

for a suitable $\varepsilon > 0$. Clearly, K^{τ} is translation invariant and a tail event. For each $\xi \in K^{\tau}$ and every $x \in r\mathbb{Z}^d$, there is an $l(\xi, x) < \infty$ such that

$$\text{agr}_{k, x}(\xi, \tau) > 1 - \varepsilon \tag{3.6}$$

whenever $k > l(\xi, x)$. In other words, $\xi \in K^{\tau}$ resembles τ (but not uniformly) when considering the frequency of agreement over large enough regions around any $x \in r\mathbb{Z}^d$.

To obtain now a condition in terms of stochastic-geometric properties, we consider for each couple of configurations $(\sigma, \sigma') \in \Omega_{\Lambda_n} \times \Omega_{\Lambda_n}$ the set of sites $x \in \Lambda_n$ for which $(\sigma(x), \sigma'(x)) \neq (a, a)$. A path of disagreement (with respect to the ground state) joining a region $A \subset \Lambda_n$ with a region $B \subset \Lambda_n$, $A \cap B = \varnothing$, is a sequence $x_0 \in A, x_1, x_2, \ldots, x_n \in B$ of ordered nearest neighbour sites $x_j \in \Lambda_n$ for which $(\sigma(x_j), \sigma'(x_j)) \neq (a, a)$, for all $j = 1, 2, \ldots, n$. Define the event

$$\Pi_n(A, B) = \{(\sigma, \sigma') \in \Omega_{\Lambda_n} \times \Omega_{\Lambda_n} : \text{there is a path of disagreement from A to B}\} \tag{3.7}$$

Obviously, all of the above can be repeated for the cubes $\Lambda_n(x) = \Lambda_n + x$, $x \in \mathbb{Z}^d$, obtaining thus the analogues $\Pi_n^x(A, B)$ of $\Pi_n^0(A, B) = \Pi_n(A, B)$. In each of the cubes we consider $V_n(x)$, and define for $\xi \in \Omega_r$

$$\xi_{k,x}(y) = \begin{cases} \xi(y) & \text{if } y \in V_k(x) \\ a & \text{if } y \in r\mathbb{Z}^d \setminus V_k(x) \end{cases} \tag{3.8}$$

Definition 3.1. We say that μ_β^τ is a *stable low temperature phase* if there exist real numbers $C = C(\beta) < \infty$ and $m = m(\beta) > 0$, with $m(\beta) \to \infty$ as $\beta \to \infty$, such that for each $x \in \mathbb{Z}^d$

$$\mu_{\beta, \Lambda_n(x)}^{\tau, \xi_{k,x}} \times \mu_{\beta, \Lambda_n(x)}^{\tau, \xi_{k,x}}(\Pi_n^x(O, B)) \leqslant Ce^{-mk} \tag{3.9}$$

uniformly in n, whenever $n > k > l(\xi, x)$, $\xi \in K^\tau$, and where O is the set of nearest neighbour sites to the origin, and $B = \Lambda_n(x) \setminus \Lambda_k(x)$.

In typical Pirogov-Sinai situations condition (3.9) is satisfied but we are not concerned with its proof here.

We will now investigate what are the consequences of condition (3.9) of the stable low temperature phases for decimated Gibbs measures. Let us consider

$$\nu_{\beta,n}^\tau(\xi) = \sum_{\sigma \in \Omega_{\Lambda_n}} \mu_{\beta,\Lambda_n}^\tau(\sigma)\, I(\sigma = \xi \text{ on } V_n) \tag{3.10}$$

for $\xi \in \Omega_{V_n}$.

An important object to look at is the relative energy

$$h_n^\tau(\xi) = \log \frac{\nu_{\beta,n}^\tau(\xi)}{\nu_{\beta,n}^\tau(\xi^0)} \tag{3.11}$$

obtained in the situation of having $\xi \in \Omega_{V_n}$ changed into ξ^0. Here

$$\xi^0(x) = \begin{cases} \xi(x) & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases} \tag{3.12}$$

Next we take a sequence of volumes $D_k \subset r\mathbb{Z}^d$ constructed iteratively by adding single sites $D_k = D_{k-1} \cup \{a_k\}$, $a_k \in r\mathbb{Z}^d$, with the properties $|a_k| \geqslant |a_{k-1}|$, $a_1 = 0$, $D_0 = \varnothing$ and $D_1 = \{0\}$, and $|a_k| \leqslant |x|$, for all $x \in r\mathbb{Z}^d$ with $|x| \geqslant |a_{k-1}|$. Typically, for $k \simeq n^d$, D_k will have covered the set V_n. We also put

$${}^k\xi(x) = \begin{cases} \xi(x) & \text{if } x \in D_k \\ a & \text{if } x \in r\mathbb{Z}^d \setminus D_k \end{cases} \tag{3.13}$$

It is useful to think of D_k as almost coinciding with some $V_{k'}$, and then ${}^k\xi \simeq \xi_{k'}$, $k \simeq (k')^d$.

By telescoping (3.11) we get

$$h_n^\tau(\xi) = \sum_{k=1}^{s(n)} [h_n^\tau({}^k\xi) - h_n^\tau({}^{k-1}\xi)] \tag{3.14}$$

where $s(n)$ is determined by the relation $D_{s(n)} = V_n$. Note that ${}^0\xi = \tau$ and $h_n^\tau(\tau) = 0$.

By putting

$$\Phi_k^n(\xi) := h_n^\tau({}^k\xi) - h_n^\tau({}^{k-1}\xi) \tag{3.15}$$

and by defining the function (remember $\tau(x) = a$, for all x)

$$f_x^{\tau,\xi}(\sigma) = \exp\left(-\beta \sum_{y:\,|y-x|=1} [U(a, \sigma(y)) - U(\xi(x), \sigma(y))] \right) \tag{3.16}$$

we arrive at the following

Lemma 3.1. For $k > 2$ we have

$$\Phi_k^n(\xi) = \log[1 + \psi_k^n(\xi)\, \mu_{\beta,\Lambda_n}^{\tau,{}^k\xi}(f_0^{\tau,\xi}; f_{a_k}^{\tau,\xi})] \tag{3.17}$$

where

$$1/\psi_k^n(\xi) = \mu_{\beta,\Lambda_n}^{\tau,{}^k\xi}(f_0^{\tau,\xi})\, \mu_{\beta,\Lambda_n}^{\tau,{}^k\xi}(f_{a_k}^{\tau,\xi}) \in [e^{-\beta c d}, e^{\beta c d}] \tag{3.18}$$

for some finite $c > 0$ (d is the dimension of the lattice).

Here $\mu(f; g) = \mu(fg) - \mu(f)\, \mu(g)$ is the truncated correlation function.

Proof. We write

$$\Phi_k^n(\xi) = \log \frac{\nu_{\beta,n}^\tau({}^k\xi)}{\nu_{\beta,n}^\tau(({}^k\xi)^0)} \frac{\nu_{\beta,n}^\tau({}^{k-1}\xi)}{\nu_{\beta,n}^\tau({}^k\xi)} \frac{\nu_{\beta,n}^\tau({}^k\xi)}{\nu_{\beta,n}^\tau(({}^{k-1}\xi)^0)} \tag{3.19}$$

Moreover we have

$$\frac{\nu_{\beta,n}^\tau({}^k\xi)}{\nu_{\beta,n}^\tau(({}^k\xi)^0)} = \frac{\sum_{\sigma \in \Omega_n} \mu_{\beta,\Lambda_n}^\tau(\sigma) I(\sigma = {}^k\xi \text{ on } V_n)}{\sum_{\sigma \in \Omega_n} \mu_{\beta,\Lambda_n}^\tau(\sigma)\, I(\sigma = ({}^k\xi)^0 \text{ on } V_n)} \tag{3.20}$$

$$I(\sigma = ({}^k\xi)^0 \text{ on } V_n) = I(\sigma = {}^k\xi \text{ on } V_n \backslash \{0\})\, I(\sigma = a \text{ in } 0) \tag{3.21}$$

Hence, by an application of the finite volume DLR equations, (3.17) readily follows. ∎

Furthermore, one can show that this truncated correlation function is controlled by condition (3.9):

Lemma 3.2. For all ξ we have

$$|\mu^{\tau,\xi}_{\beta,\Lambda_n}(f^{\tau,\xi}_0; f^{\tau,\xi}_{a_k})| \leqslant \text{const}(\beta)\ \mu^{\tau,\xi}_{\beta,\Lambda_n} \times \mu^{\tau,\xi}_{\beta,\Lambda_n}[\Pi_n(O, \Lambda_n \backslash \Lambda_k)] \quad (3.22)$$

where $\text{const}(\beta) = 2e^{4\beta \|U\|_\infty}$, $\|U\|_\infty = \sup_{v_1, v_2 \in S} |U(v_1, v_2)|$.

Proof. By making use of the ideas in ref. [3], or adapting almost word for word the proof of Proposition 2 in ref. [30]. ∎

Hence we have

Proposition 3.1. If μ^τ_β is a stable low temperature phase, then there exist some numbers $0 < c(\beta) < \infty$, and $\delta(\beta) > 0$ with $\delta(\beta) \to \infty$ as $\beta \to \infty$, such that

$$|\Phi^n_k(\xi)| \leqslant c(\beta)\, e^{-\delta(\beta) k} \quad (3.23)$$

for all $\xi \in K^\tau_l(0)$ whenever $k > l(\xi, k)$.

The same analysis can be repeated for relative energies with respect to changes in the configuration at a site $x \neq 0$, $x \in r\mathbb{Z}^d$.

We want to mention that in ref. [18] a similar condition on truncated correlation functions was considered to conclude that the transformed measure is a Gibbs measure. The difference from our results is that there a uniform control in ξ has been applied.

By using Sullivan's method (Theorem 1 in ref. [38]), we can construct then an interaction $\{U_B\}_{B \in r\mathbb{Z}^d}$ so that $\lim_n \nu^\tau_{\beta,n} = \nu^\tau_\beta$ (always assuming that this limit exists) is a weakly Gibbsian measure with respect to $K = K^\tau$ and potential $\{U_B\}$. This method was applied in ref. [30] to find that the projection to the line of the + phase in the low temperature two dimensional Ising model is a weakly Gibbsian measure. In ref. [1] we find the proof in the case of decimation applied for the same measure. An interesting question (answered on the affirmative in ref. [1]) is whether the constructed potential is the same for different ground states $\tau^{(i)}$.

We believe that the scenario sketched here for decimation remains essentially unchanged for other (renormalization group-) transformations: If we have a pure phase in the Pirogov-Sinai situation, the transformed measure is at worst weakly Gibbsian. This would take care of a substantial part of the Griffiths-Pearce-Israel pathologies described in ref. [10].

4. THERMODYNAMIC FUNCTIONS AND VARIATIONAL PRINCIPLE

In this section we present results on thermodynamic functions of weakly Gibbsian measures, partly of a general validity and partly for the class of weakly Gibbsian measures obtained from transformations of Gibbs measures. We start with the general results and indicate when we switch to the particular class. Throughout this section the inverse temperature will be included in the Hamiltonian. Also, we denote by $\mathscr{P}(K)$ the set of translation invariant weakly Gibbsian measures for the set K, which contains configurations "typical" for an interaction $\{U_B\}$.

The energy function

$$E(\sigma) = \sum_{B \ni 0} \frac{1}{|B|} U_B(\sigma) \tag{4.1}$$

is defined for each $\sigma \in K$. For $\nu \in \mathscr{P}(K)$ we define the energy density as $e(\nu) = \nu(E)$

Proposition 4.1. Suppose $\nu \in \mathscr{P}(K)$ such that $\nu(b_0) < \infty$. Then

$$e(\nu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \sum_{B \subset \Lambda} \nu(U_B) = \lim_{\Lambda} \frac{1}{|\Lambda|} \nu(H_\Lambda) \tag{4.2}$$

(limits are taken along increasing sequences of cubes).

Proof. By translation invariance, for each $\Lambda \in \mathscr{E}$

$$\begin{aligned} e(\nu) &= \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{B \ni x} \frac{1}{|B|} \nu(U_B) \\ &= \frac{1}{|\Lambda|} \sum_{B \subset \Lambda} \nu(U_B) + \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{B \ni x \\ B \cap \Lambda^c \neq \varnothing}} \frac{1}{|B|} \nu(U_B) \end{aligned} \tag{4.3}$$

The last term is bounded by

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{B \ni x \\ B \cap (\Delta + x)^c \neq \varnothing}} \nu(|U_B|) + \frac{1}{|\Lambda|} \sum_{\substack{x \in \Lambda \\ \Delta + x \not\subset \Lambda}} \sum_{B \ni x} \nu(|U_B|) \tag{4.4}$$

for every $\Delta \in \mathscr{E}$. The second term in (4.4) is further bounded by

$$\frac{1}{|\Lambda|} |\{x \in \Lambda : \Delta + x \not\subset \Lambda\}| \, \nu(b_0) \tag{4.5}$$

which goes to zero, for fixed Δ, as $\Lambda \to \mathbb{Z}^d$. On the other hand, the first term in (4.4) is identical to

$$\sum_{\substack{B \ni 0 \\ B \cap \Delta^c \neq \varnothing}} \nu(|U_B|) \tag{4.6}$$

and it goes to zero by taking Δ arbitrarily large. ∎

Proposition 4.2. Take $\nu \in \mathscr{P}(K)$ with $\nu(b_0) < \infty$, and suppose that $\tau \in K$ satisfies the "reference" condition (2.2). Then

$$e(\nu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \nu(H^{\tau}_{\Lambda}) \tag{4.7}$$

Proof. By comparison with the proof of Proposition 4.1, we must now check that

$$\frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sum_{\substack{B \ni x \\ B \cap \Lambda^c \neq \varnothing}} \nu(|U_B(\sigma_{\Lambda} \tau_{\Lambda^c})|) \to 0 \tag{4.8}$$

By using (2.2), however, this reduces to the proof of Proposition 4.1. ∎

Proposition 4.3. Take $\nu \in \mathscr{P}(K)$, and suppose that there exists a function $\Psi: \mathbb{N} \to \mathbb{R}^+$, monotonically decreasing and $\sum_{n=1}^{\infty} \Psi(n)\, n^{d-1} < \infty$. Assume, moreover, that

$$|I_{V,W}(\sigma)| \leqslant \tfrac{1}{2} \sum_{\substack{x \in V \\ y \in W}} \Psi(|x-y|)[f_x(\sigma) + f_y(\sigma)] \tag{4.9}$$

for all $V, W \in \mathscr{E}$, $V \cap W = \varnothing$, with a non-negative measurable function $f_x(\sigma) = f(\theta_x \sigma)$, $\nu(f_x) < \infty$, for all $x \in \mathbb{Z}^d$. Then we have

$$e(\nu) = \lim_{\Lambda} \frac{1}{|\Lambda|} \nu(H^{\tau}_{\Lambda}) \tag{4.10}$$

for every $\tau \in K$ for which

$$\sup_{x} \sum_{y \in \mathbb{Z}^d} \Psi(|x-y|)\, f_y(\tau) < \infty \tag{4.11}$$

Proof. Take $\sigma \in K$. We have

$$I_{A, A^c}(\sigma) = \sum_{\substack{B \cap A \neq \varnothing \\ B \cap A^c = \varnothing}} U_B(\sigma) \tag{4.12}$$

By comparing with the proof of Proposition 4.1 it is clear that we must show that

$$\frac{1}{|A|} \nu(|I_{A, A^c}(\sigma_A \tau_{A^c})|) \to 0 \tag{4.13}$$

The assumption allows us to split off this expression into two terms. First consider

$$\frac{1}{2\,|A|} \sum_{\substack{x \in A \\ y \in A^c}} \Psi(|x-y|)\, \nu(f) \tag{4.14}$$

This is going to zero since we can split off this part further into

$$\sum_{x \in A} \Psi(|x-y|)\, \nu(f) = \sum_{x \in V} \Psi(|x-y|)\, \nu(f) + \sum_{x \in A \setminus V} \Psi(|x-y|)\, \nu(f) \tag{4.15}$$

where $\text{dist}(V, A^c) \to \infty$ as $A \to \mathbb{Z}^d$ and $|V|/|A| \leqslant 1$.

Secondly, we must look at

$$\frac{1}{2\,|A|} \sum_{\substack{x \in A \\ y \in A^c}} \Psi(|x-y|)\, f_y(\tau) \tag{4.16}$$

But this can be treated in exactly the same way if we have indeed hypothesis (4.11) on $\tau \in K$. ∎

Remark 4.1. The point Proposition 4.3 makes is that we have here a situation analogous with the case of unbounded spins, where the hypothesis on $I_{V,W}$ is referred to as a regularity condition on the interaction. It is reasonable to expect that these conditions (as formulated in Proposition 4.3) will be satisfied in interesting examples. We think of $f_x(\sigma)$ as $f_x(\sigma) = l(x, \sigma)$, $\sigma \in K$ (see (3.6)).

For $\nu \in \mathscr{P}(K)$, the specific (negative) entropy

$$s(\nu) = \lim_{A} \frac{1}{|A|} \sum_{\sigma \in \Omega_A} \nu(\sigma) \log \nu(\sigma) \tag{4.17}$$

exists in $[-\log|S|, 0]$. For two probability measures ν_1 and ν_2 on $(\Omega_\Lambda, \mathscr{F}_\Lambda)$, the relative entropy of ν_1 with respect to ν_2 is

$$S_\Lambda(\nu_1 \mid \nu_2) = \sum_{\sigma \in \Omega_\Lambda} \nu_1(\sigma) \log \frac{\nu_1(\sigma)}{\nu_2(\sigma)} \tag{4.18}$$

(The usual conventions apply). We always have $S_\Lambda \geqslant 0$, and $S_\Lambda(\nu_1 \mid \nu_2) = 0$ if and only if $\nu_1 = \nu_2$.

Consider now a *bona fide* Gibbs measure μ on $(\Omega, \mathscr{F})$ with respect to a Hamiltonian $\mathscr{H}$. The finite volume specification for a fixed boundary condition $a \in S$ is

$$\mu_\Lambda^a(\sigma) = \frac{1}{Z_\Lambda^a} \exp[-\mathscr{H}_\Lambda^a(\sigma)] \tag{4.19}$$

Let $T_{\Lambda, V}$ be a probability kernel from $(\Omega_\Lambda, \mathscr{F}_\Lambda)$ to $(\Omega_V, \mathscr{F}_V)$, such that Λ and V are subsets of lattices having the same dimensionality, and with $V \subset \Lambda$, and $r|V| = |\Lambda|$. Define $\nu_V^a = \mu_\Lambda^a T_{\Lambda, V}$, and let $\nu_V = \mu_\Lambda T$, where μ_Λ is the restriction of μ to $(\Omega_\Lambda, \mathscr{F}_\Lambda)$. By using the monotonicity of the relative entropy with respect to a (stochastic) transformation T we are led to

Proposition 4.4. Take a sublattice $\mathscr{V}$ of $\mathbb{Z}^d$ with $\dim \mathscr{V} = d$, and pick a van Hove volume $V \subset \mathscr{V}$. Then we have

$$\lim_V \frac{1}{|V|} S_V(\nu_V \mid \nu_V^a) = 0 \tag{4.20}$$

In particular, this holds when $\nu \in \mathscr{P}(K)$ is a weakly Gibbsian measure obtained from a Gibbs measure μ (i.e., $\nu = \mu T$) under a transformation T (for example, a decimation as in the previous section).

Take T corresponding to a decimation transformation:

$$T_{\Lambda_n, V_n}(\sigma, \xi) = \prod_{x \in V_n} I(\sigma(x) = \xi(x)) \tag{4.21}$$

for $\sigma \in \Omega_{\Lambda_n}$ and $\xi \in \Omega_{V_n}$. We take $\nu = \mu T$, a weakly Gibbsian state obtained from a Gibbs measure μ. As already seen, $\nu_\Lambda = \mu_\Lambda T_{\Lambda, V} = (\mu T)_\Lambda$, but moreover

$$\gamma_\Lambda^\tau = \mu_\Lambda(\cdot \mid \xi = \tau \text{ on } r\mathbb{Z}^d \backslash V)\, T_{\Lambda, V} \tag{4.22}$$

Here $\mu_\Lambda(\cdot \mid \xi = \tau \text{ on } r\mathbb{Z}^d \backslash V)$ is the restriction to $(\Omega_\Lambda, \mathscr{F}_\Lambda)$ of μ conditional of finding the configuration $\tau \in K$ outside V (but on the sites of the decimated lattice $r\mathbb{Z}^d$). Therefore, by using again the monotonicity of relative entropies, we obtain

Proposition 4.5. Let V be a van Hove volume of $\mathscr{V}$ ($\dim \mathscr{V} = d$), and $\tau \in K$. Then we have

$$\lim_V \frac{1}{|V|} S_V(\nu_V \mid \gamma_V^\tau) = 0 \tag{4.23}$$

Now we turn to investigating the pressure (free energy). The set K and the potential $\{U_B\}$ are given as in Section 2.

Proposition 4.6. Suppose $\nu = \mu T$, the decimation of a Gibbs measure μ, is weakly Gibbsian with respect to $(K, \{U_B\})$. Furthermore, we assume that for $\nu \in \mathscr{P}(K)$ we have $\nu(b_0) < \infty$. Then the limiting specific free energy (or pressure)

$$p = \lim_V \frac{1}{|V|} \log Z_V^\tau \tag{4.24}$$

exists and it is independent of $\tau \in K$, whenever $e(\nu) = \lim_V 1/|V| \, \nu(H_V^\tau)$ exists. The conditions in Propositions 4.2 or 4.3 provide sufficient conditions.

Proof. For each $V \in \mathscr{E}$ and $\tau \in K$ we can use the weakly Gibbsian measure ν to write

$$S_V(\nu \mid \gamma_V^\tau) = S_V(\nu) + \nu(H_V^\tau) + \log Z_V^\tau \tag{4.25}$$

On the other hand

$$\lim_V \frac{1}{|V|} S_V(\nu \mid \gamma_V^\tau) = 0 \tag{4.26}$$

for every $\tau \in K$, hence the proposition follows. ∎

Remark 4.2. From the proof of Proposition 4.6 it is evident that weakly Gibbsian states fulfilling the conditions above also satisfy the variational principle

$$-p = s(\nu) + e(\nu) \tag{4.27}$$

We have, however, no converse statement of any sort. We expect that for a converse statement extra conditions will be needed on $(K, \{U_B\})$ for producing a richer theory.

Finally, it is interesting to observe that $s(\nu^+ \mid \nu^-) \neq 0$ (non-vanishing specific relative entropy) where ν^+ and ν^- are the projections to the line of the pure phases μ^+ and μ^- of the planar Ising model (see ref. [36]). We believe, however, that ν^+ and ν^- are weakly Gibbsian with respect to *the same* potential, even though further decimations yield two Gibbs measures for two inequivalent potentials, as shown in ref. [26].

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